Problem 1 – Counting Practice. Each of the following problems listed is equivalent to another listed problem. Find the matches!

(a) Find the number of ways to select m people out of n to be on a committee.

(b) Find the number of ways to distribute n identical candies to m distinguishable children. Note that children don't have to get exactly 1 candy — they can get 0 or multiple candies.

(c) Find the number of ways for n friends to order from a menu of m items if each friend orders 1 dish and every item must be ordered at least once. We only care about what arrives to the table, not which particular friend ordered each dish. That is, we only care about the total number of each item ordered.

(d) Find the number of ways to select integers $x_1 \ldots x_m$ to fill in $x_1 + x_2 + \ldots + x_m = n$, where $x_i \geq 0$.

(e) Find the number of ways to select $m - 1$ people out of $n - 1$ people to be on a committee.

(f) Find the number of paths from (0, 0) to (m, n-m) where each step is either up or to the right.

Problem 2 – Combinatorial Proofs. For each of the following, provide a combinatorial proof by showing that each side of the expression counts the same thing. They can all be done by a committee-forming metaphor, but you may select a different metaphor if you'd prefer.

(a) Show that
$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

(b) Show that
$$\binom{2n}{2} = \binom{n}{2} + \binom{n}{2} + n^2.$$

(c) Show that
$$\binom{n+1}{k} = (n+1) \cdot \binom{n}{k-1} \cdot \frac{1}{k}.$$

Problem 3 – Probability Partitions. Match five of the following statements to the values of $\Pr(A)$, $\Pr(B)$, or 1, and identify the remaining statement that cannot be matched to one of these values. Assume that $0 < \Pr(A), \Pr(B) < 1$.

(a) $\Pr(A \cap B) + \Pr(A^c \cap B)$

(b) $\Pr(A|B) + \Pr(A^c|B)$

(c) $\Pr(A|B^c)\Pr(B^c) + \Pr(A|B)\Pr(B)$

(d) $\Pr(A|B)\Pr(B) + \Pr(A^c|B)\Pr(B)$

(e) $\Pr(A|B) + \Pr(A|B^c)$

(f) $\Pr(A \cap B) + \Pr(A \cap B^c)$

Problem 4 – Bayes' Theorem. You have one fair coin, and one biased coin which lands Heads with probability $\frac{3}{4}$. You pick one of the coins at random and flip it three times. It lands Heads all three times. Given this information, what is the probability that the coin you picked is the fair one?

Hint: Assign what you observed to an event A and what you want to know the probability of to an event F, then compute $\Pr(F|A)$ by Bayes' Theorem.