Problem 1 – Independence. Suppose we pull two cards from a standard deck of cards (with 52 cards, 26 black cards, and 4 Aces). Let X be the number of Aces and Y be the number of black cards in our two cards.

(a) How many possible outcomes are there when pulling two cards (at the same time) from the deck?

There are $\binom{52}{2}$ outcomes.

(b) Compute $\Pr(X = 2)$.

The probability of pulling two Aces is $\frac{\binom{4}{2}}{\binom{52}{2}}$

(c) Compute $\Pr(Y = 2)$.

The probability of pulling two black cards is $\frac{\binom{26}{2}}{\binom{52}{2}}$

(d) Compute $\Pr(X = 2, Y = 2)$.

The probability of pulling two black Aces is $\frac{1}{\binom{52}{2}}$.

(e) Are X and Y independent?

We check if $\frac{1}{\binom{52}{2}} = \frac{\binom{4}{2}}{\binom{52}{2}} \cdot \frac{\binom{26}{2}}{\binom{52}{2}}$ and get that these variables are not independent!

Problem 2 – Indicators. Suppose we pull 5 cards from a standard 52-card deck containing 4 Aces. Let X be the number of Aces in my hand. We'll compute $E(X)$ two ways.

(a) What is the probability that the first card I pull is an Ace?

The probability is $\frac{4}{52} = \frac{1}{13}$ — we are equally likely to pull any card, and there are 4 Aces among 52 cards.

(b) What is the probability that the second card I pull is an Ace? Note that we have two conditions: when the first card I pull is an Ace, and when the first card I pull is not an Ace.

The probability that the first card was an Ace is $\frac{1}{13}$, and the probability that it wasn't an Ace is thus $\frac{12}{13}$. If the first card was an Ace, I have 3 Aces left among 51 remaining cards, so the probability that the second card is an Ace is $\frac{3}{51}$. On the other hand, if the first card wasn't an Ace, all 4 Aces are left so the probability that the second card is an Ace is $\frac{4}{51}$. Putting this together we have:

$$\frac{3}{51} \cdot \frac{1}{13} + \frac{4}{51} \cdot \frac{12}{13} = \frac{1}{13}$$

(c) Use a counting argument to show that the probability that the ith card (for $1 \leq i \leq 5$) is an Ace is $\frac{1}{13}$.

There are $\frac{52!}{47!}$ ways of arranging the 5 cards in order. There are 4 possible Aces I could get in position i, and $\frac{51!}{47!}$ ways of filling in the other cards around the Ace. So using naive probability, we have:

$$\frac{4 \cdot \frac{51!}{47!}}{\frac{52!}{47!}} = \frac{1}{13}$$

(d) Compute $E(X)$ using indicator variables, where $I_i = 1$ if an Ace is the ith card in my hand (for $1 \leq i \leq 5$).

We have that $X = I_1 + I_2 + I_3 + I_4 + I_5$, so by linearity of expectation, $E(X) = E(I_1) + E(I_2) + E(I_3) + E(I_4) + E(I_5)$. Each of these Bernoulli random variables is 1 with probability $\frac{1}{13}$ (and 0 with probability $\frac{12}{13}$). So we have $E(X) = 5 \cdot \frac{1}{13} = \frac{5}{13}$.

(e) Now we'll work towards a different indicator variable. What is the probability that the Ace of Spades is in my 5-card hand?

There are $\binom{52}{5}$ ways of making my 5-card hand. The number of ways of making a 5-card hand with the Ace of Spades in it is $\binom{51}{4}$, since we need to choose the other 4 cards from the 51 remaining options. So the probability is

$$\frac{\binom{51}{4}}{\binom{52}{5}} = \frac{5}{52}.$$

This extends to any of our 4 Aces!

(f) Compute $E(X)$ using indicator variables, where $I_i = 1$ if the ith Ace (of the Ace of Spaces, Ace of Hearts, Ace of Diamonds, and Ace of Clubs) is in my hand (for $1 \leq i \leq 4$).

We have that $X = I_1 + I_2 + I_3 + I_4$, so by linearity of expectation, $E(X) = E(I_1) + E(I_2) + E(I_3) + E(I_4)$. Thus we have

$$E(X) = \frac{5}{52} + \frac{5}{52} + \frac{5}{52} + \frac{5}{52} = \frac{4 \cdot 5}{52} = \frac{5}{13}.$$