Problem 1 Two cards are drawn, without replacement, from a standard, well-shuffled deck of cards. Let A be the event that the first card is a heart, and B be the event that the second card is red. Compute the following probabilities.

(a) $P(A \cap B)$

We use a decision process to count the number of outcomes where the first card is a heart and the second card is red: we have 13 choices for the first card, and 25 choices for the second card (the number of red cards minus one, since we drew a red card for the first card), so there are $13 \cdot 25$ outcomes in $A \cap B$. Since there are $52 \cdot 51$ outcomes total, we have

$$P(A \cap B) = \frac{13 \cdot 25}{52 \cdot 51} = \frac{25}{204}.$$

(b) $P(A \mid B)$

$P(B) = \frac{1}{2}$, since the 2 colors are equally likely (we can also use a decision process to compute this!). Therefore

$$P(A \mid B) = \frac{25/204}{1/2} = \frac{25}{102}.$$

(c) $P(B \mid A)$

$P(A) = \frac{1}{4}$, since the 4 suits are equally likely (we can also use a decision process to compute this!). Therefore

$$P(B \mid A) = \frac{25/204}{1/4} = \frac{25}{51}.$$

Problem 2 Consider a coin biased to show Heads with probability $\frac{3}{4}$, and suppose we flip the coin n times. What is the probability that we get exactly k Heads, for $0 \leq k \leq n$?

Each specific arrangement of k Heads among $n - k$ Tails has probability

$$(\frac{3}{4})^k (\frac{1}{4})^{n-k}.$$

There are $\binom{n}{k}$ of these arrangements, therefore our probability is

$$\binom{n}{k} (\frac{3}{4})^k (\frac{1}{4})^{n-k}.$$

Problem 3 You're testing out a machine learning classifier that is designed to identify whether an image contains a cat. The classifier correctly identifies 90% of cat images as containing cats. However, it also incorrectly labels 15% of non-cat images as containing cats. You run it on a dataset where 10% of images actually contain cats. What is the probability that an image labeled as a cat actually contains a cat?

Let L be the event that an image is labeled as containing a cat and C be the event that an image contains a cat.

Then we are given $\Pr(L \mid C^c) = 0.15$ which gives us $\Pr(L^c \mid C^c) = 0.85$. We are also given $\Pr(L \mid C) = 0.90$ which gives us $\Pr(L^c \mid C) = 0.10$. Therefore $\Pr(L) = \Pr(L \mid C^c) \Pr(C^c) + \Pr(L \mid C) \Pr(C) = 0.15 \cdot 0.9 + 0.9 \cdot 0.10 = 0.225$.

Using Bayes' Theorem, we have

$$\Pr(C \mid L) = \frac{\Pr(L \mid C) \Pr(C)}{\Pr(L)} = \frac{0.90 \cdot 0.10}{0.225} = 0.4.$$