Problem 1 Use the stars and bars technique to count the number of outcomes in the following scenarios:

(a) How many ways are there to order 10 scoops of ice cream from 5 flavors?

This is our classic stars and bars example — we have 10 stars (scoops) and 4 bars (to separate the scoops into 5 flavors), giving us

$$\binom{14}{10}$$

(b) How many ways are there to place 10 indistinguishable balls in 5 bins?

For each (indistinguishable) ball, we choose one of 5 bins to place it in (with replacement — we're allowed to put multiple balls in the same bin).

So we have 10 stars (balls) and 4 bars (to separate the balls into 5 bins), giving us

$$\binom{14}{10}$$

(c) From a bin with 5 distinguishable balls, you draw a ball 10 times, with replacement. How many possibilities are there for the number of times each ball is drawn?

Each of our 10 draws gets categorized as the ball (of the 5 distinguishable balls) drawn. We only care about the number of times each ball was drawn, but not the order.

So we have 10 stars representing each draw and 4 bars (to separate each draw into one of the 5 balls), giving us

$$\binom{14}{10}$$

(d) How many solutions are there to the equation $a + b + c + d + e = 10$ where $a, b, c, d, e \geq 0$?

We can think of a, b, c, d, and e as our options that we want to choose exactly 10 of, in some combination. So we have 10 stars each representing a unit of size 1 and 4 bars (to separate the units into the five variables), giving us

$$\binom{14}{10}$$

Problem 2 Each of the following problems requires at least one of the tricks from class: PIE, Complementary Counting, and Stars and Bars. Some use multiple!

(a) How many positive integers less than 100 are a multiple of 3 or 5?

We can use PIE here. Denote by

$$A = \{x : (3|x) \wedge (0 < x < 100)\},$$
$$B = \{x : (5|x) \wedge (0 < x < 100)\}.$$

Then we know

$$A \cap B = \{x : (3|x) \wedge (5|x) \wedge (0 < x < 100)\} = \{x : (15|x) \wedge (0 < x < 100)\}.$$

We are looking for $|A \cup B|$. By PIE, we know that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

A has 33 members (3 to 99), B has 19 members (5 to 95), and $A \cap B$ has 6 members (15 to 90), so

$$|A \cup B| = 33 + 19 - 6 = \boxed{46}.$$

(b) How many positive integers less than 100 are not a multiple of 3 or 5?

The simplest way to solve this problem is to notice that it's the direct complement of the previous problem! There are 99 positive integers less than 100, and 46 of them are multiples of 3 or 5, so $99 - 46 = \boxed{53}$ are not.

(c) A painter has been commissioned to paint 5 houses. The paint shop has three different colors to choose from, and the painter buys a bucket of paint per house. How many different paint orders could be made? (Two paint orders are considered the same if they have the same amount of each color of paint, regardless of order)

This problem is equivalent to distributing 5 indistinguishable balls (buckets to buy) among 3 labeled bins (colors).

By stars and bars, there are $\boxed{\binom{7}{5}}$ configurations.

(d) A painter has been commissioned to paint 5 houses. They have three different colors to choose from, but at least one pair of adjacent houses must have the same color. How many ways are there to paint the houses?

It's difficult to deal with all of the possibilities for adjacent houses here, so the easiest way around it is to take the complement instead. In that case, we'd like to know the number of ways to paint the houses such that *no two adjacent houses have the same color*. Then there is a decision process through which we can simply paint the houses one at a time, making sure that each house is not the same as the previously-painted one:

- Choose a color for the first house (3 options)
- Choose a color for the second house that is not the same as the previous house's color (2 options)
- Choose a color for the third house that is not the same as the previous house's color (2 options)

- Choose a color for the fourth house that is not the same as the previous house's color (2 options)
- Choose a color for the fifth house that is not the same as the previous house's color (2 options)

This gives us $3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 48$ ways to paint the houses such that no two adjacent houses have the same color.

On the other hand, with no restrictions, there are 3 color choices for each house so $3^5 = 243$ ways to paint the houses. Thus, in $243 - 48 = \boxed{195}$ ways to paint the houses where at least one pair of adjacent houses has the same color.

(e) Ten distinguishable trees need to be planted in a row. However, if the tallest tree and the shortest tree are next to one another, the shortest tree will not get any sunlight. How many ways are there to plant the trees so that the tallest tree and shortest tree are not adjacent?

Again, the adjacent condition is difficult to deal with. It's much easier to think about the case where the two trees *are* adjacent and then use **complementary counting**. To count the number of ways to plant the trees where the tallest and shortest are adjacent, we follow the decision process:

- First, place the tallest and shortest trees. Since they must be adjacent to each other, there are 9 pairs of adjacent places for them to be in, and for each pair there are 2 options for the order of the trees, for a total of 18 options of how to place the tallest and shortest trees.
- Next, place the remaining 8 trees in the 8 remaining spots - there are 8! ways to do this, since the trees are distinguishable.

Thus, there are $18 \cdot (8!)$ configurations where the tallest and shortest trees are adjacent. There are 10! configurations in total, leaving $\boxed{10! - 18 \cdot (8!)} = 2903040$ configurations where they are not adjacent.

Note: This could have also been counted directly, by counting the number of ways to place two non-adjacent trees, and then placing the rest. There are $\binom{10}{2}$ positions for placing two trees, so $2 \cdot \binom{10}{2}$ ways to place the two trees overall (because the trees are distinguishable so order matters). There are 18 ways to place the two trees such that they are adjacent, so there are $2 \cdot \binom{10}{2} - 18 = 72$ ways to place the two trees such that they are not adjacent. There are 8! ways to place the other trees, so we have $72 * 8! = 2903040$ total ways.

(f) Ten indistinguishable trees need to be planted along a row spanning three blocks. How many ways are there to distribute the trees among the blocks?

This is another classic instance of stars and bars - there are 10 indistinguishable balls (trees) to put in one of three bins (blocks). Thus, there are $\boxed{\binom{12}{10}}$ distributions of trees.

Problem 3 Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter, three-digit arrangement is equally likely, find the probability that such a license plate will contain at least one palindrome (contains a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left, or both).

For example, each of these license plates contain a palindrome: ABA123, ABC121, ABA121.

(a) Use complementary counting (compute the number of license plates that do *not* contain a palindrome, and subtract this from the total number of license plates).

First, we count the number of non-palindrome license plates. Either three-character group is a palindrome if the first character matches the third character. Thus, our decision process for each three-character group that is not a palindrome is as follows:

- Select the first character.
- Select the second character.
- Select the third character to be different from the first.

This creates $26 \cdot 26 \cdot 25$ three-letter arrangements and $10 \cdot 10 \cdot 9$ three-number arrangements, for a total of $(26 \cdot 26 \cdot 25) \cdot (10 \cdot 10 \cdot 9)$ license plates with no palindrome.

On the other hand, there are $26^3 \cdot 10^3$ license plates in total, so the probability of having a palindrome is

$$\frac{26^3 \cdot 10^3 - (26 \cdot 26 \cdot 25) \cdot (10 \cdot 10 \cdot 9)}{26^3 \cdot 10^3} = \frac{26^2 \cdot 10^2 \cdot (26 \cdot 10 - 25 \cdot 9)}{26^3 \cdot 10^3} = \frac{260 - 225}{260} = \boxed{\frac{7}{52}}.$$

(b) Use PIE (let A be the set of license plates with letter-based palindromes and B be the set of license plates with number-based palindromes)

We consider the outcomes where there is a letter-based palindrome, a number-based palindrome, and both. A palindrome is formed by the decision process:

- Select the first character.
- Select the second character.
- Select the third character to be the same as the first.

The first two selections are entirely unconstrained, whereas the last decision only has one choice. Thus there are 26^2 letter-based palindromes, making $26^2 \cdot 10^3$ license plates with a letter-based palindrome. Similarly, there are 10^2 number-based palindromes, making $26^3 \cdot 10^2$ license plates with a number-based palindrome. However, $26^2 \cdot 10^2$ license plates have both palindromes. Therefore, the probability of selecting a license plate with a palindrome is

$$\frac{26^2 \cdot 10^3 + 26^3 \cdot 10^2 - 26^2 \cdot 10^2}{26^3 \cdot 10^3} = \frac{26^2 \cdot 10^2 \cdot (10 + 26 - 1)}{26^3 \cdot 10^3} = \frac{35}{260} = \boxed{\frac{7}{52}}.$$