Problem 1 – Set Practice. Let $S = \{1, 2, 3\}$.

(a) List the elements of the cartesian product $S \times S$.

(b) List the elements of the power set $P(S)$.

(c) We have that $S \in P(S)$. Is $S \subseteq P(S)$?

(a) $S \times S : \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

(b) $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

(c) No. We only have $S \subseteq P(S)$ if every element of S (that is, the integers 1, 2, and 3) is an element of $P(S)$. But $P(S)$ contains sets, not integers, so while the set S is an element of $P(S)$, none of the *elements* of S are elements of $P(S)$.

Problem 2 Use decision processes to compute the number of options for the following scenarios:

(a) How many two-digit **numbers** are there whose digits are different? (ex: 09 is not counted as a two-digit number, but 10 is)

(b) How many two-digit **strings** are there whose digits are different? (ex: "09" does count as a two-digit string)

(c) How many ways are there to arrange 6 people in a line?

(a) Decision process: choose from 9 digits (1 through 9) for the first digit, and from 9 digits (everything except what was just chosen) for the second digit. So we have $\boxed{9 \cdot 9 = 81}$ numbers whose digits are different.

(b) Decision process: choose from 10 digits (0 through 9) for the first digit, and 9 digits (everything except what was just chosen) for the second digit, so we have $\boxed{10 \cdot 9 = 90}$ strings whose digits are different.

(c) Decision process: select the first person to be in line from all six people, select the second person from the remaining five, select the third person from the remaining four, select the fourth person from the remaining three, select the fifth from the remaining two, and place the last remaining person in the sixth spot. This gives $\boxed{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} - 720$ options.

Problem 3 The student council has 6 first-year students, 4 second-year students, 3 third-year students, and 8 fourth-year students.

(a) How many ways are there to form a committee consisting of one student from each year?

Our decision process is to select one student from the first-years (6 possibilities), then one from the second-years (4 possibilities), then one from the third-years (3 possibilities), then one from the fourth-years (8 possibilities). Thus, our decision process reveals $\boxed{6 \cdot 4 \cdot 3 \cdot 8}$ total possibilities.

(b) How many ways are there to form a committee consisting of two students from each year?

Similar to last problem, our decision process is to select two students from the first-years, then two from the second-years, then two from the third-years, then two from the fourth-years. In each case, the order in which the students are selected in each committee does not matter. Thus, our decision process reveals

$$\boxed{\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{3}{2} \cdot \binom{8}{2}}$$

total possibilities.

(c) How many ways are there to form a committee with the same number of students from each year, regardless of how many there are?

Our decision process will first decide how many students we'll pick from each year, then determine how many ways there are to form committees of that size. Because there are only three third-year students, there cannot be more than three students per year in the committee. The cases for 1, 2, and 3 students per year are entirely non-overlapping, so we can simply follow the logic from the prior two problems and add the three-student case:

$$\boxed{6 \cdot 4 \cdot 3 \cdot 8 + \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{3}{2} \cdot \binom{8}{2} + \binom{6}{3} \cdot \binom{4}{3} \cdot \binom{3}{3} \cdot \binom{8}{3}}$$

(d) How many ways are there to form a committee consisting of two underclass (1st/2nd year) and two upperclass (3rd/4th year) students?

Our decision process simply first selects two underclass students and then selects two upperclass students. There are 10 underclass students and 11 upperclass students, for a total of

$$\boxed{\binom{10}{2} \cdot \binom{11}{2}}$$

possibilities.

(e) Each member of the student council shakes the hand of each other member exactly once. How many handshakes occur?

Each handshake occurs between two different people on the council. Thus, the number of handshakes is exactly the number of ways to select two different people on the council, where the order in which they are selected doesn't matter. This gives us

$$\boxed{\binom{21}{2}}$$

handshakes.

(f) Each grade must elect a President and a Vice President (who must be different people) out of that grade's members on the student council. How many ways are there to assign these roles across the student council?

A grade with n students has $n(n-1)$ ways to choose the President and Vice President. So we have:

$$\boxed{6 \cdot 5 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 8 \cdot 7}$$

ways of assigning these roles.

(g) Each member of the student council is assigned to either the subcommittee for student life, academics, or clubs. How many possible assignments of members to subcommittees are there? (Subcommittees are allowed to be empty, but every member must be on a committee. No member can be in more than one subcommittee.)

One decision process we can use is to simply go in order by member, assigning each member to a subcommittee. At each step, the member has three possibilities for subcommittees, and there are 21 different members in total, which means there are $\boxed{3^{21}}$ possibilities.

(h) A new rule stipulates that members do not need to serve on a subcommittee, but they may serve on one of the three committees. How many possible assignments of members to subcommittees are there?

Along with the three committees, we introduce a fourth option, ”no committee”. Again, we go member by member, assigning each to one of these options. At each step, the member has four options, and there are 21 different members in total, so there are $\boxed{4^{21}}$ possibilities.