Problem 1 Your local ATM only dispenses 2 and 5 dollar bills, but the farmers market requires exact change! Prove that for any $n \geq 4$, n dollars can be made up of 2 and 5 dollar bills.

We proceed by strong induction on n. Let $P(n)$ represent the predicate that we can obtain n dollars with 2 and 5 dollar bills.

First, we show two base cases:

- P(4) = 2 + 2
- P(5) = 5

For our inductive hypothesis, let $k \in \mathbb{N}$ be arbitrary with $k \geq 5$ and assume that $P(i)$ is true for $4 \leq i \leq k$. We show that $P(k+1)$ is true.

Note that $P(k-1)$ is true (because $k \geq 5$), and adding a two-dollar bill to the bills from $P(k-1)$ gives us $k+1$ dollars. Therefore $P(k+1)$ is true, and we have completed our inductive step. Therefore we can make any amount n out of 2 and 5 dollar bills for all $n \geq 4$.

Problem 2 The chicken nugget problem is a real math problem, originally posed when McNuggets came in packs of 6, 9, and 20 in the UK. Show that for every $n \geq 44$, we can get n chicken nuggets from packs of 6, 9, and 20 nuggets.

We proceed by strong induction on n.

We first show six base cases:

- $44 = 20 + 6 \cdot 4$
- $45 = 9 \cdot 3 + 6 \cdot 3$
- $46 = 20 \cdot 2 + 6$
- $47 = 20 + 9 \cdot 3$
- $48 = 6 \cdot 8$
- $49 = 20 \cdot 2 + 9$

For our inductive hypothesis, let $k \in \mathbb{N}$ be arbitrary with $k \geq 49$ and assume for all $44 \leq j \leq k$ that $j = 20a + 9b + 6c$ for some $a, b, c \in \mathbb{N}$ (that is, j can be made up from packs of 20, 9, or 6 chicken nuggets). Consider $k + 1$. Since $k \geq 49$, $k - 5 \geq 44$ and therefore by inductive hypothesis $k - 5 = 20a + 9b + 6c$ for some $a, b, c \in \mathbb{N}$. Then

$$k + 1 = k - 5 + 6 = (20a + 9b + 6c) + 6 = 20a + 9b + 6(c + 1).$$

This completes the inductive step. Therefore, we have shown that $\forall n \geq 44$, n nuggets can be obtained from packs of 6, 9, and 20.

Note: we also could do a shorter proof, as seen in class, claiming that we just add a pack of 6 to $P(n - 5)$!

Problem 3 I claim that I can show, by strong induction, that for every $n \geq 0$, $5n = 0$.

Find the error in my proof:

Base case: Let $n = 0$. Then $5 \cdot 0 = 0$, so our base case holds.

Inductive hypothesis: Let $k \in \mathbb{N}$ be arbitrary. Assume that $5j = 0$ for all $0 \leq j \leq k$.

Inductive step: We show that $5(k+1) = 0$. Note that $k+1 = i+j$ for some $i, j < k+1$. Therefore $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$ (by our inductive hypothesis). Therefore, for every $n \geq 0$, $5n = 0$. $\square$

The base case is true, and I can't argue with the inductive hypothesis (we're allowed to assume that!). However, not every $k+1$ can be split into some $i+j$ where $i, j < k+1$ (consider $k+1 = 1$.) So the logic of the inductive step does not hold for all $k+1$!