Problem 1 – Another Sum. Prove that the sum of the first n odd numbers is n^2. Namely, prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2.$$

We proceed by induction on n.

Base Case: Let $n = 0$. Then $2 \cdot 0 + 1 = 1$ and $(0+1)^2 = 1$ so $\sum_{i=0}^{n}(2i+1) = (n+1)^2$ holds for $n = 0$.

Inductive Hypothesis: Let $k \in \mathbb{N}$ be arbitrary and assume that $\sum_{i=0}^{k}(2i+1) = (k+1)^2$.

Inductive Step: We aim to show that $\sum_{i=0}^{k+1}(2i+1) = (k+2)^2$.
Note that

$$\sum_{i=1}^{k+1}(2i+1) = \sum_{i=1}^{k}(2i+1) + (2(k+1)+1).$$

By the Inductive Hypothesis, this is equal to

$$\begin{aligned}
\sum_{i=1}^{n}(2i+1) + (2(n+1)+1) &= (n+1)^2 + 2n + 2 + 1 \\
&= n^2 + 2n + 1 + 2n + 2 + 1 \\
&= n^2 + 4n + 4 \\
&= (n+2)^2,
\end{aligned}$$

as desired.
Thus, we have shown that $\forall n \ \sum_{i=0}^{n}(2i+1) = (n+1)^2$.

Problem 2 – Induction Inequality. Prove that $n < 2^n$ for $n \in \mathbb{N}$.

We proceed by induction on n.

Base Case: Let $n = 0$. Then $0 < 2^0 = 1$ so $n < 2^n$ holds for $n = 0$.

Inductive Hypothesis: Let $k \in \mathbb{N}$ be arbitrary and assume that $k < 2^k$.

Inductive Step: We aim to show that $k + 1 < 2^{k+1}$.
By our inductive hypothesis, we have that $k + 1 < 2^k + 1$.

Note that $1 < 2^k$, so $2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

Thus, we have shown that $\forall n \in \mathbb{N}\ n < 2^n$.

Problem 3 – Find the error in this proof of $P(n)$.

$P(n)$: Every set of n distinct lines in a plane (where $n \geq 2$), no two of which are parallel, meet at the same point.

Proof: We proceed by induction on n.

Our base case is $P(2)$. Any 2 lines in a plane that are not parallel meet at a common point by definition of parallel lines, so $P(2)$ is true.

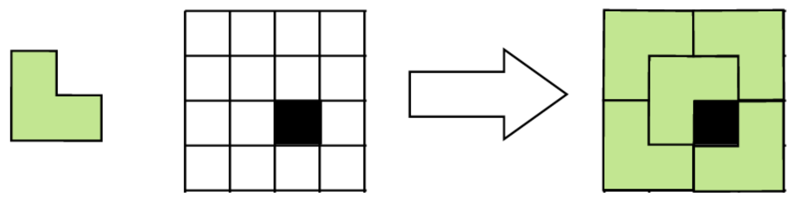
Next, let k be arbitrary in $\mathbb{N}$ and assume that $P(k)$ is true — every set of k distinct lines in a plane, no two of which are parallel, meet at a common point. We show that $P(k+1)$ is true. Consider a set of $k+1$ distinct lines in a plane, no two of which are parallel. By the inductive hypothesis, the first k of these lines meet at a common point; call this point p_1. Similarly, the last k of these lines meet at a common point; call this point p_2. Note that, apart from the first and last line in the plane, all other lines cross both p_1 and p_2. We will show that p_1 and p_2 must be the same point, and thus all $k+1$ lines meet at the same point.

Recall that two points define a line. If p_1 and p_2 were different points, all lines containing both p_1 and p_2 would be the same line. This contradicts our assumption that all lines are distinct. Thus, both p_1 and p_2 are the same point, and we conclude that this point is the intersection of all $k+1$ lines. Therefore, we have shown that if $P(k)$ is true then $P(k+1)$ is true, completing our proof. $\square$

Solution: The inductive step requires that $k \geq 3$ so that *two* lines will be in the first and second group, crossing both p_1 and p_2, creating the contradiction. Thus the inductive step **does not hold** from $P(2)$ to $P(3)$.

Problem 4 Consider a square $2^n \times 2^n$ grid and imagine placing L-shaped domino pieces of area 3 on this grid. This grid has a special condition: one square is always removed. A **tiling** of the grid is a way of placing the pieces so that no two pieces overlap and all grid cells are covered (except for the removed square).

An example of a valid tiling using 5 L-shaped dominoes is shown below:



Note: every domino piece has the shape described above and covers exactly three squares, every input grid has exactly one removed (black) square, but the location of the removed square may change between input grids.

Prove by induction on n that every $2^n \times 2^n$ grid with one square removed can be tiled successfully using L-shaped dominoes, for $n \geq 1$.

Solution: See Rosen page 348.

Problem 5 – Water Balloon Fight. There are $2n + 1$ people at a water balloon fight, each with one water balloon. When the fight starts, each person throws the balloon at their nearest neighbor (assume, for the sake of the problem, that there are no equally-near neighbors!). Show that, for $n > 0$, at least one person is not hit. (Hint: Start by looking at the two people closest to each other.)

Solution: See Rosen page 347.