Problem 1 – Grab Bag. This is a problem to get some practice and familiarity with the topics from number theory. Full proofs are not necessary.

(a) For what numbers a does $-1|a$?

(b) For what numbers a does $a|-1$?

(c) True or False: for positive integers a and b, if $a|b$ then $a \leq b$.

(d) Find q and r satisfying the division theorem for $n = -15$, $d = 7$.

(e) Use Euclid's algorithm to find $\gcd(483, 91)$.

(f) Write out the elements of the set $S = \{p + n : \exists p \in \mathbb{N}, \exists n \in \mathbb{N}, n = p^2, n < 10\}$.

(a) There must exist $n(-1) = a$. Note that for any a, $n = -a$ also exists and $(-a)(-1) = a$, so a can be all natural numbers.

(b) There must exist $n(a) = -1$. Note that this is only possible if $a = \pm 1$.

(c) True: because a and b are both positive and $an = b$, n must also be positive. But if n is positive, then $a \leq b$.

(d) We need q and r such that $-15 = 7q + r$ with $0 \leq r < 7$. This is satisfied by $q = -3, r = 6$.

(e)

$$\begin{aligned} \gcd(483, 91) &= \gcd(91, 28) \\ &= \gcd(28, 7) \\ &= \gcd(7, 0) \\ &= 7 \end{aligned}$$

(f) For n to be in S, it must be both a square and less than 10. Thus, the only values of n are $1, 4, 9$. For these values, the corresponding p are $1, 2, 3$, so $S = \{2, 6, 12\}$.

Problem 2 – GPT4 vs. Number Theory. In this problem, I have asked GPT4 to prove a number theory statement. Identify everywhere a mistake is present in the solution.

Problem: Prove that if 27 divides XYZ, then 27 also divides YZX. (Here, XYZ is the three-digit number with **digits** X, Y, and Z, not $X \times Y \times Z$.)

GPT4 Solution:

(1) To prove this, we will use the fact that a number is divisible by 27 if and only if the sum of its digits is divisible by 27.

(2) Let the three-digit number XYZ be represented as $100X + 10Y + Z$. Here, X, Y, and Z are the digits of the number and each ranges from 0 to 9. We are given that 27 divides XYZ, i.e., 27 divides $(100X + 10Y + Z)$.

(3) Consider the number YZX, which can be represented as $100Y + 10Z + X$. We need to show that 27 also divides $(100Y + 10Z + X)$.

(4) We start by rearranging the expression for XYZ:

$$100X + 10Y + Z = 99X + X + 10Y + Z = 99X + (X + 10Y + Z)$$

(5) Since 27 divides $(100X + 10Y + Z)$, it must also divide $(X + 10Y + Z)$ because $99X$ is a multiple of 27 (as $99 = 3 * 33$, and 27 is a multiple of 3).

(6) Now, consider the expression for YZX:

$$100Y + 10Z + X = 99Y + Y + 10Z + X = 99Y + (Y + 10Z + X).$$

(7) We already know that 27 divides $(X + 10Y + Z)$. We can rearrange $(X + 10Y + Z)$ as follows:

$$X + 10Y + Z = X + 10(Y + Z) - 9Z = X - 9Z + 10(Y + Z).$$

(8) Similarly, rearranging $(Y + 10Z + X)$:

$$Y + 10Z + X = Y + 10(Z + X) - 9X = Y - 9X + 10(Z + X).$$

(9) Since $(X + 10Y + Z)$ and $(Y + 10Z + X)$ are cyclic permutations of the digits, they are congruent modulo 9 (as subtracting 9 times a digit does not affect divisibility by 9). Therefore, if 27 divides one of these sums, it divides both sums.

(10) Thus, since 27 divides $(X + 10Y + Z)$, it must also divide $(Y + 10Z + X)$. Therefore, 27 divides $100Y + 10Z + X$, which means 27 divides the number YZX.

(11) This concludes the proof that if 27 divides a three-digit number XYZ, then 27 also divides YZX.

(1) While this is true for 3 and 9, this rule does not hold for 27.

(5) $99X$ is not always a multiple of 27, for example when $X = 1$.

(9) Cyclic permutations are not in the scope of this class, but the first sentence is correct here. However, this does not mean that these numbers have the same divisibility by 27.

Problem 3 Now it's your turn! Prove that if 27 divides XYZ, then 27 also divides YZX. *Hint: GPT-4 did do some things right! In particular, $XYZ = 100X + 10Y + Z$ is a helpful place to start.*

We proceed by direct proof. We are given that $27|XYZ$, so there exists some m such that $100X + 10Y + Z = 27m$. Thus we can write $1000X + 100Y + 10Z = 270m$. We know that $YZX = 100Y + 10Z + X$, so rearranging we find $YZX - X = 100Y + 10Z$. Substituting into the prior equation, we find

$$\begin{aligned} 1000X + 100Y + 10Z &= 270m \\ 1000X + (YZX - X) &= 270m \\ YZX + 999X &= 270m \\ YZX &= 270m - 999x \\ YZX &= 27(10m - 37x), \end{aligned}$$

so YZX is divisible by 27 as well.