Problem 1 – Counting Practice. Each of the following problems listed is equivalent to another listed problem. Find the matches!

(a) Find the number of ways to select m people out of n to be on a committee.

This is a classic binomial coefficient example, giving us $\binom{n}{m}$ — order doesn't matter, and we don't have replacement. **Matches with (f)**.

(b) Find the number of ways to distribute n identical candies to m distinguishable children. Note that children don't have to get exactly 1 candy — they can get 0 or multiple candies.

This is a stars and bars problem. We have n identical stars (the candies) that need to be assigned to m different children, requiring $m-1$ bars. Therefore we have $\binom{n+m-1}{n}$ ways to do this. **Matches with (d)**.

(c) Find the number of ways for n friends to order from a menu of m items if each friend orders 1 dish and every item must be ordered at least once. We only care about what arrives to the table, not which particular friend eats each dish. That is, we only care about the total number of each item ordered.

Since every item must be ordered at least once, we actually only have to count the number of ways to order the remaining $n-m$ additional items from a menu of m items. We use stars and bars, with $n-m$ stars (representing the orders) and $m-1$ bars (assigning these orders to menu items). So we have $\binom{(n-m)+(m-1)}{n-m} = \binom{n-1}{n-m} = \binom{n-1}{m-1}$ ways to do this. **Matches with (e)**.

(d) Find the number of ways to select integers $x_1 \ldots x_m$ to fill in $x_1 + x_2 + \ldots + x_m = n$, where $x_i \geq 0$.

This is one of our stars and bars examples, with n stars and $m-1$ bars, giving $\binom{n+m-1}{n}$. **Matches with (b)**.

(e) Find the number of ways to select $m-1$ people out of $n-1$ people to be on a committee.

This is a classic use of the binomial coefficient, giving us $\binom{n-1}{m-1}$. **Matches with (c)**.

(f) Find the number of paths from (0, 0) to (m, n-m) where each step is either up or to the right.

We need to take m steps to the right and $n-m$ steps up, totaling $m+n-m=n$ steps in our path. Of those n steps, we choose exactly m to go to the right and go up for the rest of the steps. Therefore there are $\binom{n}{m}$ paths. **Matches with (a)**.

Problem 2 – Combinatorial Proofs. For each of the following, provide a combinatorial proof by showing that each side of the expression counts the same thing. They can all be done by a committee-forming metaphor, but you may select a different metaphor if you'd prefer.

(a) Show that
$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Both sides count the number of ways to select a committee of r people from n people, and a subcommittee of k people from those r people. The left side selects the committee of r people first and then selects the subcommittee from those people, whereas the right side selects the subcommittee of k people first, and then fills in the rest of the committee.

(b) Show that
$$\binom{2n}{2} = \binom{n}{2} + \binom{n}{2} + n^2.$$

We are selecting two people from $2n$ total. Imagine there are n tall people and n short people. Then our two people could either be two tall people, two short people, or one of each. Then the right side says we can pick two tall people ($\binom{n}{2}$), two short people ($\binom{n}{2}$), or one of each (n^2).

(c) Show that
$$\binom{n+1}{k} = (n+1) \cdot \binom{n}{k-1} \cdot \frac{1}{k}.$$

The left side of the equation counts the number of ways to select k people from $n+1$ people. The right side counts the same by first selecting one person — there are $n+1$ ways to choose the first person. Then, there are $\binom{n}{k-1}$ ways to select the remaining $k-1$ people. But this over-counts each commitee of k people exactly k times, one per person chosen first (if we first select person A, then people B and C, that is equivalent to first selecting person B, then A and C, or first selecting person C, then A and B). Since we overcount every group k times, we divide out a factor of k.

Problem 3 – LOTP. Match five of the following statements to the values of $\Pr(A)$, $\Pr(B)$, or 1, and identify the remaining statement that cannot be matched to one of these values. Assume that $0 < \Pr(A), \Pr(B) < 1$.

(a) $\Pr(A \cap B) + \Pr(A^c \cap B) = \Pr(B)$

Because A and A^c are events that partition the sample space, the LOTP gives us that this sums to $\Pr(B)$.

(b) $\Pr(A|B) + \Pr(A^c|B) = 1$

By our definition of conditional probability, $\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$ and $\Pr(A^c|B) = \frac{\Pr(A^c\cap B)}{\Pr(B)}$. This gives us $\frac{\Pr(A\cap B)}{\Pr(B)} + \frac{\Pr(A^c\cap B)}{\Pr(B)} = \frac{\Pr(A\cap B)+\Pr(A^c\cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1$.

(c) $\Pr(A|B^c)\Pr(B^c) + \Pr(A|B)\Pr(B) = \Pr(A)$

By the definition of conditional probability, this is $\Pr(A \cap B^c) + \Pr(A \cap B)$. By the law of total probability, this is $\Pr(A)$.

(d) $\Pr(A|B)\Pr(B) + \Pr(A^c|B)\Pr(B) = \Pr(B)$

By the definition of conditional probability, this is $\Pr(A \cap B) + \Pr(A^c \cap B)$. By the law of total probability, this is $\Pr(B)$.

(e) $\Pr(A|B) + \Pr(A|B^c)$

This is the probability of A given B plus the probability of A given B^c... however, this does not evaluate to the probability of A! This is because we don't know the relative probabilities of B and B^c. By the definition of conditional probability, this is $\frac{\Pr(A\cap B)}{\Pr(B)} + \frac{\Pr(A\cap B^c)}{\Pr(B^c)}$, but we can't simplify any further.

(f) $\Pr(A \cap B) + \Pr(A \cap B^c) = \Pr(A)$

Just like in (a), because B and B^c are events that partition the sample space, the LOTP gives us that this sums to $\Pr(A)$.

Problem 4 – Bayes' Theorem. You have one fair coin, and one biased coin which lands Heads with probability $\frac{3}{4}$. You pick one of the coins at random and flip it three times. It lands Heads all three times. Given this information, what is the probability that the coin you picked is the fair one?

Hint: Assign what you observed to an event A and what you want to know the probability of to an event F, then compute $P(F|A)$ by Bayes' Theorem.

By the hint, we let A be the event that the chosen coin lands Heads three times and F be the event that we picked the fair coin. We want to find $P(F|A)$.

By Bayes Theorem, $\Pr(F|A) = \frac{\Pr(A|F)\Pr(F)}{\Pr(A)} = \frac{\Pr(A|F)\Pr(F)}{\Pr(A|F)\Pr(F)+\Pr(A|F^c)\Pr(F^c)}$.

Computing each of these values:

$$\Pr(A|F) = 0.5^3$$

$$\Pr(F) = 0.5$$

$$\Pr(A|F^c) = 0.75^3$$

$$\Pr(F^c) = 0.5$$

Therefore $\Pr(F|A) = \frac{0.5^3 \cdot 0.5}{0.5^3 \cdot 0.5 + 0.75^3 \cdot 0.5} = \frac{8}{35} \approx 0.23$.