Welcome to Discrete Math!

We'll start in a few minutes. In the meantime:

- Grab 2 worksheets from the front of class
- Read through the course website (in your email)
- Welcome survey posted on Ed

Course Learning Objectives

We'll learn the mathematical tools and concepts relevant to computer scientists (who work with data that is ***discrete***, as opposed to continuous)

Develop skills for **rigorous mathematical reasoning** (proof writing and reading)

Learn core concepts of mathematical fields relevant to CS:

- **Logic** — mathematical reasoning with statements that are either true or false
- **Number theory** — integers and their properties, with a particular focus on prime numbers
- **Combinatorics** — the study of how to count things, specifically large but finite sets of elements
- **Discrete probability** — quantifying the likelihood that random events will occur
- **Graph theory** — the study of graphs, a data model composed of vertices and edges used to represent relationships between objects

Defining Course Policies *(more details on course website)*

Active, collaborative learning in every lecture

Your first attempt is not your defining attempt!

- All homeworks can be re-submitted (same problems) for full credit
  - Must make a **real attempt** on your first try!
  - Must **reflect** on what your original solution was missing
- Exams 1 and 2 will have retakes (new problems) for partial credit

Asking for help is part of the learning process!

- Credit for asking/answering questions on Ed, coming to office hours
  - Required for homework 1
  - Extra credit added to exams, up to 3% (attendance counts too)

Course Logistics

5 problem sets (plus resubmissions for full credit) **25% of grade**

3 unit exams (plus two retakes for partial credit) **75% of grade**

- Up to 3% of extra exam credit can be earned by **attendance/participation**

AI Policy

General collaboration/resource policy: don't solicit full solutions (or check your full solution)

AI isn't always right: on Thursday, we'll do an exercise finding the error in a GPT-generated proof

The point of homework is to prepare for the exams, where no outside resources will be allowed — take advantage of this time!

Logic + Proofs

Lecture 1

Propositional Logic

The building blocks of proofs!

Definition (Proposition)

A **proposition** p is a statement that is **either true or false.**

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Chicago is capital of Illinois.

Springfield is the capital of Illinois.

Either Chicago or Springfield is the capital of Illinois.

It is June.

June is in the winter.

June is in the summer.

Definition (Proposition)

A **proposition** *p* is a statement that is **either true or false.**

Chicago is capital of Illinois. **F**

Springfield is the capital of Illinois. **T**

Either Chicago or Springfield is the capital of Illinois. **T**

It is June. **T**

June is in the winter. **F**

June is in the summer. **T**

Definition (Proposition)

A **proposition** *p* is a statement that is **either true or false.**

Chicago is capital of Illinois. **F**

Springfield is the capital of Illinois. **T**

Either Chicago or Springfield is the capital of Illinois. **T**

It is June. **T**

June is in the winter. **F**

June is in the summer. **T**

Not a proposition: *let's go to the lake, should we have ice cream for dinner?*

Logical Connectives

Alone, propositions don't let us claim much. Connectives allow us to join one or more propositions together and denote relationships between propositions.

Definition (Negation)

For a proposition p, $\neg p$ is the **negation** of p, pronounced "not p". The proposition $\neg p$ is defined to be **true when p is false and false when p is true**.

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$\neg p$: Chicago is not the capital of Illinois.

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p	$\neg p$
T	
F	

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p	$\neg p$
T	F
F	T

Definition (And)

For propositions p and q, $p \wedge q$ is the **conjunction** of p and q, pronounced "p and q". The proposition $p \wedge q$ is defined to be true if **both p and q are true**.

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p: Chicago is the capital of Illinois.

q: Springfield is the capital of Illinois.

$p \wedge q$: *Chicago is the capital of Illinois* ***and*** *Springfield is the capital of Illinois.*

Definition (And)

For propositions p and q, $p \land q$ is the **conjunction** of p and q, pronounced "p and q". The proposition $p \land q$ is defined to be true if **both *p* and *q* are true**.

p: Chicago is the capital of Illinois. **F**

q: Springfield is the capital of Illinois. **T**

p $\land$ *q: Chicago is the capital of Illinois **and** Springfield is the capital of Illinois.* **F**

Definition (And)

For propositions p and q, $p \land q$ is the **conjunction** of p and q, pronounced "p and q". The proposition $p \land q$ is defined to be true if **both *p* and *q* are true**.

p	q	$p \land q$
T	T	
T	F	
F	T	
F	F	

Definition (And)

For propositions p and q, $p \land q$ is the **conjunction** of p and q, pronounced "p and q". The proposition $p \land q$ is defined to be true if **both p and q are true**.

p	q	$p \land q$
T	T	T
T	F	
F	T	
F	F	

Definition (And)

For propositions p and q, $p \land q$ is the **conjunction** of p and q, pronounced "p and q". The proposition $p \land q$ is defined to be true if **both *p* and *q* are true**.

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition (Or)

For propositions p and q, $p \lor q$ is the **disjunction** of p and q, pronounced "p or q". The proposition $p \lor q$ is defined to be true **if p is true or if q is true**.

Definition (Or)

For propositions p and q, $p \vee q$ is the **disjunction** of p and q, pronounced "p or q". The proposition $p \vee q$ is defined to be true **if p is true or if q is true**.

$p \vee q$: *Chicago is the capital of Illinois **or** Springfield is the capital of Illinois.* T

Definition (Or)

For propositions p and q, $p \vee q$ is the **disjunction** of p and q, pronounced "p or q". The proposition $p \vee q$ is defined to be true **if *p* is true or if *q* is true**.

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

Definition (Or)

For propositions p and q, $p \vee q$ is the **disjunction** of p and q, pronounced "p or q". The proposition $p \vee q$ is defined to be true **if p is true or if q is true**.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition (Or)

For propositions p and q, $p \vee q$ is the **disjunction** of p and q, pronounced "p or q". The proposition $p \vee q$ is defined to be true **if p is true or if q is true**.

p	q	$p \vee q$
T	T	T
T	F	T
F		
F		

Note: this is not the exclusive or!

Definition (Conditional)

For propositions p and q, the proposition $p \to q$ is the **implication** of p and q, pronounced "if p then q". We call p the hypothesis and q the conclusion.

Definition (Conditional)

For propositions p and q, the proposition $p \rightarrow q$ is the **implication** of p and q, pronounced “if p then q”. We call p the hypothesis and q the conclusion.

We can think of $p \rightarrow q$ as a *promise* rather than a causal statement:

p: If you finish your homework…

q: … then I will buy you ice cream.

Definition (Conditional)

For propositions p and q, the proposition $p \rightarrow q$ is the **implication** of p and q, pronounced “if p then q”. We call p the hypothesis and q the conclusion.

$p \rightarrow q$: If you finish your homework (p), then I will buy you ice cream (q).

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

Definition (Conditional)

For propositions p and q, the proposition $p \to q$ is the **implication** of p and q, pronounced "if p then q". We call p the hypothesis and q the conclusion.

$p \to q$: If you finish your homework (p), then I will buy you ice cream (q).

p	q	$p \to q$
T	T	T
T	F	F
F	T	
F	F	

Definition (Conditional)

For propositions p and q, the proposition $p \rightarrow q$ is the **implication** of p and q, pronounced "if p then q". We call p the hypothesis and q the conclusion.

$p \rightarrow q$: If you finish your homework (p), then I will buy you ice cream (q).

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition (Biconditional)

For propositions p and q, the proposition $p \leftrightarrow q$ is the **bidirectional implication** of p and q, pronounced "p if and only if q". We call p the hypothesis and q the conclusion.

$p \rightarrow q$: **If and only if** you finish your homework (p), then I will buy you ice cream (q).

Definition (Biconditional)

For propositions p and q, the proposition $p \leftrightarrow q$ is the **bidirectional implication** of p and q, pronounced "p if and only if q". We call p the hypothesis and q the conclusion.

$p \rightarrow q$: **If and only if** you finish your homework (p), then I will buy you ice cream (q).

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Definition (Biconditional)

For propositions p and q, the proposition $p \leftrightarrow q$ is the **bidirectional implication** of p and q, pronounced "p if and only if q". We call p the hypothesis and q the conclusion.

$p \rightarrow q$: **If and only if** you finish your homework (p), then I will buy you ice cream (q).

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order of Operations

We can string together propositions and connectives to model more complex assertions, but we'll need to define our order of operations.

Order of Operations

In order of precedence: $\neg$ (not), $\wedge$ (and), $\vee$ (or)

Left to right: $\rightarrow$ / $\leftrightarrow$ (if / if and only if)

Does $\neg p \wedge q$ mean $(\neg p) \wedge q$ *or* $\neg(p \wedge q)$ *?*

Order of Operations

In order of precedence: $\neg$ (not), $\wedge$ (and), $\vee$ (or)

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Does $\neg p \wedge q$ mean $(\neg p) \wedge q$ *or* $\neg(p \wedge q)$?

Order of Operations

In order of precedence: $\neg$ (not), $\wedge$ (and), $\vee$ (or)

Left to right: $\rightarrow$ / $\leftrightarrow$ (if / if and only if)

Does $\boldsymbol{p} \rightarrow \boldsymbol{q} \leftrightarrow \boldsymbol{r}$ mean $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \leftrightarrow \boldsymbol{r}$ *or* $\boldsymbol{p} \rightarrow (\boldsymbol{q} \leftrightarrow \boldsymbol{r})$ *?*

Order of Operations

In order of precedence: $\neg$ (not), $\wedge$ (and), $\vee$ (or)

Left to right: $\rightarrow$ / $\leftrightarrow$ (if / if and only if)

Does $\boldsymbol{p \rightarrow q \leftrightarrow r}$ mean $\boldsymbol{(p \rightarrow q) \leftrightarrow r}$ *or* $\boldsymbol{p \rightarrow (q \leftrightarrow r)}$?

Compound Truth Tables

Let's compute the truth table of $\neg p \land q$.

p	q	$\neg p$	$\neg p \land q$
T	T		
T	F		
F	T		
F	F		

Compound Truth Tables

Let's compute the truth table of $\neg p \land q$.

p	q	$\neg p$	$\neg p \land q$
T	T	F	
T	F	F	
F	T	T	
F	F	T	

Compound Truth Tables

Let's compute the truth table of $\neg p \wedge q$.

p	q	$\neg p$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Compound Truth Tables

Let's see whether $\neg(p \to q)$ is the same as $\neg p \to \neg q$.

p	q	$p \to q$	$\neg(p \to q)$	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T					
T	F					
F	T					
F	F					

Compound Truth Tables

Let’s see whether $\neg(p \rightarrow q)$ is the same as $\neg p \rightarrow \neg q$.

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	F	T	T	T

Compound Truth Tables

(from worksheet)

Yes, $((p \wedge q) \to r)$ is the same as $(p \to (q \to r))$!

Definition (Logical Equivalence)

Two propositions p and q are said to be **equivalent** if they have the same truth tables. We write $\boldsymbol{p \equiv q}$.

Basic Equivalences

$p \land q \equiv q \land p$ *(Commutativity of ∧)*

$p \lor q \equiv q \lor p$ *(Commutativity of ∨)*

$(p \land q) \land r \equiv p \land (q \land r)$ *(Associativity of ∧)*

$(p \lor q) \lor r \equiv p \lor (q \lor r)$ *(Associativity of ∨)*

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ *(Distributive law)*

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$ *(Distributive law)*

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$ *(Biconditional equivalence)*

These don't need to be memorized, but are worth trying to understand!

De Morgan's Laws

$\neg (p \land q) \equiv \neg p \lor \neg q$

$\neg (p \lor q) \equiv \neg p \land \neg q$

These two famous equivalences are fairly intuitive:

I'm not hungry **and** tired ≡ I'm not hungry **or** I'm not tired.

I'm not hungry **or** tired ≡ I'm not hungry **and** I'm not tired.

Additional Equivalences

$p \wedge \mathrm{T} \equiv p$ and $p \vee \mathrm{F} \equiv p$ (*Identity laws*)

$p \vee \mathrm{T} \equiv \mathrm{T}$ and $p \wedge \mathrm{F} \equiv \mathrm{F}$ (*Domination laws*)

$p \vee p \equiv p$ and $p \wedge p \equiv p$ (*Idempotence of* $\vee$ *and* $\wedge$)

$p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q)$ (*Absorption laws*)

$p \vee \neg p \equiv \mathrm{T}$ and $p \wedge \neg p \equiv \mathrm{F}$ (*Negation laws*)

$\neg\neg p \equiv p$ (*Double negation*)

$\neg \mathrm{T} \equiv \mathrm{F}$ and $\neg \mathrm{F} \equiv \mathrm{T}$ (*Tautology or Contradiction Negation*)

These don't need to be memorized, but are worth trying to understand!

Implication Equivalence

$\boldsymbol{p \rightarrow q \equiv \neg p \lor q}$

If you do your homework, then I will buy you ice cream.

You don't do your homework or I buy you ice cream.

Equivalence of the Contrapositive

$p \to q \equiv \neg q \to \neg p$

If you do your homework, then I will buy you ice cream.

If I didn't buy you ice cream, then you didn't do your homework.

You'll show this on the homework!

Predicate Logic

We need this to prove universal truths!

Predicate Logic

Propositional logic allows us to claim specific truths:

My dog is brown and white with spots.

Predicate logic gives us the tools to claim universal truths:

All dogs are good dogs.

To start, we need to define a **domain:** the set to which all of our objects belong.

Domain: dogs!

Domains

We define the following **domains**/sets:

- the set of **integers** $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- the set of **natural numbers** $\mathbb{N} = \{0, 1, 2, ...\}$
- the set of **real numbers** $\mathbb{R}$
- the set of **rational numbers** $\mathbb{Q}$
- the set of **complex numbers** $\mathbb{C}$

Predicates

We can define **predicates** to represent statements about objects in our domain.

For example, let $E(x)$ be the predicate representing the boolean statement "x is an even number".

$E(1)$ = **F,** $E(2)$ = **T,** $E(3)$ = **F**…

Predicates

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For example, let $E(x)$ be the predicate representing the boolean statement "x is an even number".

$E(1)$ = **F,** $E(2)$ = **T,** $E(3)$ = **F**…

Let $L(x,y)$ be the predicate representing the boolean statement "x is less than y".

$L(1,2)$ = **T**

$L(2,1)$ = **F**

Universal Quantifier

We can use **quantifiers** to express the extent of our claim about objects in a domain.

The symbol ∀ is called the **universal quantifier** and is read "for all" or “for every”.

> **For every** natural number n, $n+1 > n$.

Universal Quantifier

We can use **quantifiers** to express the extent of our claim about objects in a domain.

The symbol $\forall$ is called the **universal quantifier** and is read "for all".

For every natural number n, $n+1 > n$.

$$\forall\, n \in \mathbb{N}$$

Universal Quantifier

We can use **quantifiers** to express the extent of our claim about objects in a domain.

The symbol $\forall$ is called the **universal quantifier** and is read "for all".

> **For every** natural number n, $n+1 > n$.
>
> $\forall\ n \in \mathbb{N}, L(n, n+1)$

Example

Translate this mathematical statement into English:

$$\forall\ x \in \mathbb{Z},\ E(x) \rightarrow \neg E(x+1)$$

where $E(x)$ = "x is even".

Remember, $\mathbb{Z}$ represents the integers!

Example

Translate this mathematical statement into English:

$$\forall\ x \in \mathbb{Z},\ E(x) \rightarrow \neg E(x+1)$$

where $E(x)$ = "x is even".

Remember, $\mathbb{Z}$ represents the integers.

For every integer x, if x is even then x+1 is not even.

Existential Quantifier

The symbol $\exists$ is called the **existential quantifier** and is read "there exists".

> **There exists** a natural number *n* such that *n* is even.

Existential Quantifier

The symbol $\exists$ is called the **existential quantifier** and is read "there exists".

There exists a natural number *n* such that *n* is even.

$$\exists\ n \in \mathbb{N}, E(n)$$

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$$\forall x \; \exists y \; L(x,y)$$

$$\exists y \; \forall x \; L(x,y)$$

$$\forall x \; \exists y \; L(y,x)$$

Let $L(x,y)$ be the predicate representing the boolean statement "x is less than y".

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\forall x \; \exists y \; L(x,y)$

For every natural number x…

$\exists y \; \forall x \; L(x,y)$

$\forall x \; \exists y \; L(y,x)$

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\forall x\ \exists y\ L(x,y)$

For every natural number x, there exists a natural number y such that $x < y$.

$\exists y\ \forall x\ L(x,y)$

$\forall x\ \exists y\ L(y,x)$

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\forall x \; \exists y \; L(x,y)$

For every natural number x, there exists a natural number y such that $x < y$.

$\exists y \; \forall x \; L(x,y)$

There exists a natural number y such that for every natural number x, $x < y$.

$\forall x \; \exists y \; L(y,x)$

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\boldsymbol{\forall x\ \exists y\ L(x,y)}$

For every natural number x, there exists a natural number y such that $x < y$.

$\boldsymbol{\exists y\ \forall x\ L(x,y)}$

There exists a natural number y such that for every natural number x, $x < y$.

$\boldsymbol{\forall x\ \exists y\ L(y,x)}$

For every natural number x, there exists a natural number y such that y < x.

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\forall x \; \exists y \; L(x,y)$ T

For every natural number x, there exists a natural number y such that $x < y$.

$\exists y \; \forall x \; L(x,y)$

There exists a natural number y such that for every natural number x, $x < y$.

$\forall x \; \exists y \; L(y,x)$

For every natural number x, there exists a natural number y such that y < x.

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\forall x \; \exists y \; L(x,y)$ **T**

For every natural number x, there exists a natural number y such that $x < y$.

$\exists y \; \forall x \; L(x,y)$ **F – there is no largest natural number**

There exists a natural number y such that for every natural number x, $x < y$.

$\forall x \; \exists y \; L(y,x)$

For every natural number x, there exists a natural number y such that y < x.

Example

Let x and y be natural numbers ($x,y \in \mathbb{N}$). Let's translate the following propositions:

$\forall x\ \exists y\ L(x,y)$ T

For every natural number x, there exists a natural number y such that $x < y$.

$\exists y\ \forall x\ L(x,y)$ F – there is no largest natural number

There exists a natural number y such that for every natural number x, $x < y$.

$\forall x\ \exists y\ L(y,x)$ F — no natural number is smaller than 0

For every natural number x, there exists a natural number y such that y < x.

Theorem (De Morgan's for quantifiers)

For all predicates P,

$$\neg\, \forall x\, P \equiv \exists x\, \neg P \text{ and } \neg\, \exists x\, P \equiv \forall x\, \neg P\,.$$

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For all predicates P,

$$\neg\,\forall x\, P \equiv \exists x\, \neg P \text{ and } \neg\,\exists x\, P \equiv \forall x\, \neg P\,.$$

If $\neg\,\forall x\, P$, that means P is not true for all x.

Theorem (De Morgan's for quantifiers)

For all predicates *P*,

$$\neg \forall x\, P \equiv \exists x\, \neg P \text{ and } \neg \exists x\, P \equiv \forall x\, \neg P\,.$$

If $\neg \forall x\, P$, *that means P is not true for all x.*

So, there exists x such that P is not true… $\exists x\, \neg P$.

Theorem (De Morgan's for quantifiers)

For all predicates *P*,

$$\neg \forall x\, P \equiv \exists x\, \neg P \text{ and } \neg \exists x\, P \equiv \forall x\, \neg P\,.$$

If $\neg \forall x\, P$*, that means* P *is not true for all* x*.*

So, there exists x *such that* P *is not true…* $\exists x\, \neg P$**.**

If $\neg \exists x\, P$**,** *that means there does not exist an* x *such that* P *is true.*

Theorem (De Morgan's for quantifiers)

For all predicates *P*,

$$\neg \forall x\, P \equiv \exists x\, \neg P \text{ and } \neg \exists x\, P \equiv \forall x\, \neg P.$$

If $\neg \forall x\, P$*, that means P is not true for all x.*

So, there exists x such that P is not true… $\exists x\, \neg P$.

If $\neg \exists x\, P$*, that means there does not exist an x such that P is true.*

So, for every x, P is not true… $\forall x\, \neg P$.

On the board: our first proof!